UNIVERSITY OF TECHNOLOGY IN BRNO

Faculty of Mechanical Engineering

Institute of mechanics of solids, mechatronics and biomechanics

Syllabus

COMPUTATIONAL MECHANICS I

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1. Introduction

These lecture notes have been written for the students in the 4th year (1st year of 2nd stage) on the Faculty of Mechanical Engineering, specialization *Computational Mechanics I.* All students studying the theory of vibrations may use the text, too.

Numerical analysis has been studied since before the time of Newton. These studies were concerned with numerical procedures of approximate solutions to problems that could not be solved successfully by theoretical methods leading to analytical solutions expressed by formulas. Until the arrival of high-speed computing machines, such methods were difficult to use, and their full potential could not be realized.

With the arrival of modern computing machines the whole character of numerical analysis has changed. Iterative methods can now be used with much greater ease and effectiveness. Very large linear systems can be studied numerically. Numerical solutions of differential equations can be obtained using very small step size, or even variable step size, with thousands of steps. Today, the elaboration of efficient computational models for the analysis of the dynamic behaviour of machines and structures has became a routine task. Computers can be used to automate many engineering applications. If they are used effectively, they produce results that demonstrate an increase in the productivity and a reduction in numerical errors. Many kinds of professional software, such as ANSYS, NASTRAN, SYSNOIS, RAYNOLDS, MAPLE, MATLAB, etc., are used in order to solve the analysis of stresses and dynamic loading in frames and machines parts.

In every technical solution we have to observe the following elements of solution:

- Statement of the problem
- Finding the effective theoretical approach
- Mathematical description of the model
- Algorithm development
- Input/output design
- Choice of numerical methods
- Computer implementation
- Program development
- Program testing

We can see that the computer implementation is one of all the necessary steps only. Therefore with each type of a problem there will be a review of the theoretical ideas connected with such a problem.

2. ANALYTICAL DYNAMICS OF DISCRETE SYSTEMS

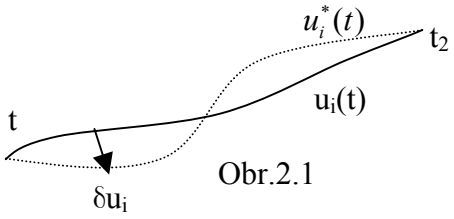
Analytical dynamics is based on the *principle of virtual work*, which is applied to the concepts of the energy and work. Since these quantities are scalars, such an approach to calculation is called *Scalar Dynamics* in opposite to the *vector dynamics.* This method serves as a very powerful tool for two main reasons:

- It considerably simplifies the analytical formulation of the motion equations for a complex mechanical system.
- It enables to apply numerical methods for the solution of both discrete and continuous systems in the most natural manner.

2.1 Principle of virtual work for a particle

Let us consider a particle of mass m_i, submitted to a force field **X** of components X_i. The dynamic equilibrium of the particle can be expressed in d'Alembert's form

$$m\ddot{u}_i - X_i = 0 \qquad i = 1,2,3$$



Obr.2.1

where u_i represents the displacement of the particle. Let us assume that the particle follows during the time interval $[t_1,t_2]$ a motion trajectory u_i^* distinct from the real one u_i (fig. 2.1). This allows us to define the virtual displacement of the particle by the relationship

$\delta u_i = u_i^* - u_i;\quad \delta u_i(t_1) = \delta u_i(t_2) = 0;$ If we multiply the dynamic equilibrium eq. (2.1.1) by the associated virtual displacement and sum the components, we get

$$\sum_{i=1}^{3}(m\ddot{u}_i - X_i)\delta u_i = 0$$

which shows that

The virtual work produced by the effective forces acting on the particle during a virtual displacement δu_i is equal to zero.

If we consider *N* particles in a system, the equation will changed to

$m_k\ddot{u}_{ik} - X_{ik} = 0 \quad for \;\; i = 1,2,3; \;\; k = 1,...N$ and the virtual work principle for the system of particles will take the form

$$\sum_{k=1}^{N}\sum_{i=1}^{3}(m_k\ddot{u}_{ik} - X_{ik})\delta u_{ik} = 0$$

It can be defined as follows:

If the virtual work equation is satisfied for any virtual displacement compatible with the kinematic constraints, the system is in a dynamic equilibrium.

The virtual work principle can be transformed into the *Hamilton's principle*:

$$\delta\int_{t_1}^{t_2}(E_k - E_p)dt = 0$$
$$\delta q(t_1) = \delta q(t_2) = 0$$

The advantage of this solution is that the expressions for velocity and displacement are used instead of acceleration and displacement. The motion equations result in the form obtained by *Lagrange*

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_s}\right) - \frac{\partial E_k}{\partial q_s} + \frac{\partial E_p}{\partial q_s} = Q_s \qquad s = 1,2,...,n$$

where E_k – kinetic energy, E_p – potential energy of conservative forces, q_s – generalised coordinates, Q_s – generalised non-conservative forces defined by

$$Q_s = \sum_{i=1}^{N} \sum_{k=1}^{3} X_{ik} \frac{\partial U_{ik}}{\partial q_s}$$

U_{ik} is the displacement of a particle as a function of q: $u_{ik}(x,t) = U_{ik}(q_1, q_2, ..., q_n, t)$.

3. VIBRATIONS

We owe the formulation of the principles describing the theory of vibration, which are applied and taught nowadays, to Lord Rayleigh. In his remarkable treatise *Theory of Sound* published in 1887 he introduced the concept of oscillations of a linear system and showed the existence of eigenmodes and eigenfrequencies for discrete as well as continuous systems. His work remains valuable in many ways although he was concerned with acoustics rather than with structural mechanics.

In general, vibration is a periodic motion and it is used to describe the oscillation in mechanical systems. In most cases, the general purpose is to prevent or attenuate the vibrations, because of their detrimental effects, such as fatigue failure of the components and generation of noise. However, there are some applications where vibrations are desirable and are usefully employed, as in vibration conveyers, vibrating sieves, etc.

The designers were the first who needed to get vibration and structural dynamic under control, because of their constant aim to minimize the stress in structures. A time passed, they had to limit the scope of their analysis and apply methods that could be handled by the available computational means.

Vibrations may be classified into three categories:

- *Free vibrations* can occur only in conservative systems where there is no friction, damping and exciting force. Here, the total mechanical energy, which is due to the initial conditions, is conserved and exchange can take place between the kinetic and potential energies.
- *External forces that excite the system cause forced vibrations*. The exciting forces supply the energy continuously to compensate for that dissipated by damping.
- *Self-excited vibrations* are periodic oscillations of the limit cycle type and are caused by some non-linear characteristics. The energy required to maintain the vibrations is obtained from a non-alternating power source. In this case, the vibrations themselves create the periodic force.

3.1 Single-degree-of-freedom systems

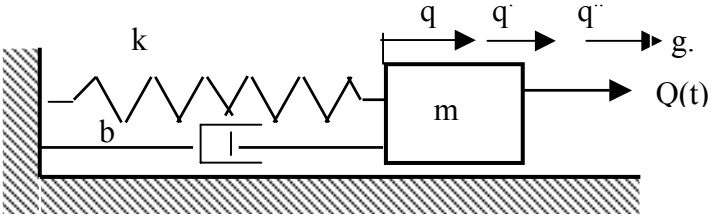


Fig. 3.1

Let us consider the model shown in fig.3.1. Displacement q is measured from the stable equilibrium position of the system, the velocity $\dot{q}$, the acceleration $\ddot{q}$ is measured positive in the positive direction of displacement. The equation of motion is: $m\ddot{q} + b\dot{q} + kq = Q(t)$

This system is one of the simplest dynamic systems in which elastic, dissipating and inertia forces interact. In torsion system the mass *m* will be replaced by mass moment of inertia *I* and the force *Q*(t) by the moment *M*(t). The solution of the differential equation of motion is composed of two parts: the solution of the homogenous equation $m\ddot{q} + b\dot{q} + kq = 0$, which gives the result

$$q_h = e^{-\delta t}(A\cos\Omega t + B\sin\Omega t) = Ce^{-\delta t}\sin(\Omega t - \varphi)$$

where:

circular frequency of damped vibration $\Omega = \sqrt{\Omega_0^2 - \delta^2}$;

natural circular frequency $\Omega_0 = \sqrt{\frac{k}{m}}$; *damping coefficient* $\delta = \frac{b}{2m}$;

The particular solution depends on the exciting force:

<u>*The force of excitation is harmonic*</u>:

$$Q(t) = Q_0 e^{\omega t}$$

$q_p = s_0 \sin(\omega t + \varphi_F - \varphi_p)$; *amplitude of forced vibration* $s_0 = \frac{Q_0}{k\sqrt{(1-\frac{\omega^2}{\Omega_0^2})^2 + (2b_r\frac{\omega}{\Omega_0^2})}}$;

phase $\varphi_p = arctg\frac{2b_r\frac{\omega}{\Omega_0}}{1-\frac{\omega^2}{\Omega_0^2}}$; *dampin gratio* $b_r = \frac{\delta}{\Omega_0}$

It is possible to introduce the *static displacement* $q_{st} = Q_0/k$. When we indicate $\frac{\omega}{\Omega_0} = \eta$ called *the tuning coefficient*, it is possible to define the *amplification coefficient* $\lambda = \frac{s_0}{q_{st}} = \frac{1}{\sqrt{(1-\eta^2)^2 + (2b_r\eta)^2}}$. This expression shows the *amplitude diagram* in the fig.3.2 .

Fig.3.3 shows the plot of relationship of phase, which is called *the phase diagram*. When b_r=0, the system is undamped. The state, when $\eta = 1$, is called resonance and $\lambda = \infty$. By the resonance the phase angle is always equal to 90^0.

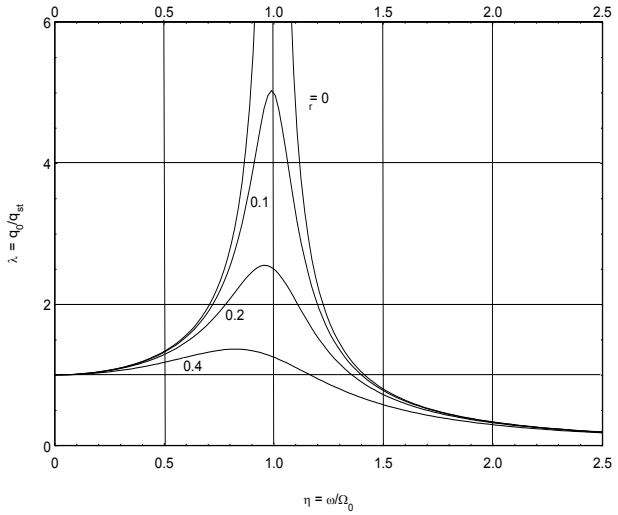


Fig. 3.2

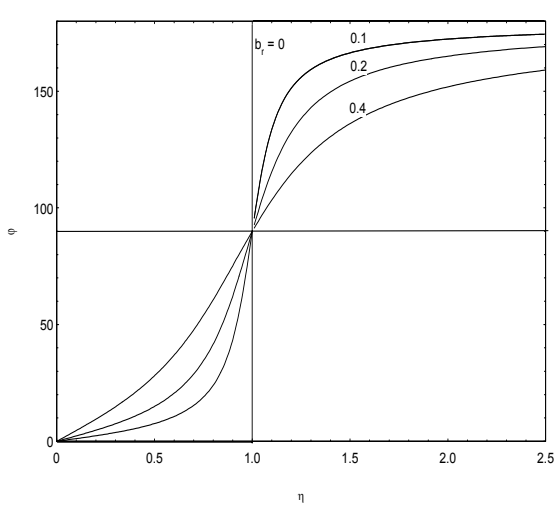


Fig. 3.3

The force is a general function of time:
The response of the system is given by following equation:

$$q_p = \frac{1}{m\Omega}\int_0^t Q(t)e^{-\delta(t-\tau)}\sin\Omega(t-\tau)d\tau$$

The force is periodic:
We assume that the exciting force is periodic. Employing Fourier Series for this force we get the differential equation of motion

$$m\ddot{q} + b\dot{q} + kq = \sum_0^\infty (F_{1i}\cos i\omega t + F_{2i}\sin i\omega t)$$

where F_{1i} and F_{2i} are the coefficients of the Fourier Series, well-known from mathematics or [5]. Since we are dealing with the vibration of a linear system, *superposition* is valid and we can consider each term of the right hand side of eq. (3.1.6) as a separate forcing function. The steady state vibration displacement is obtained by adding up the responses caused by each term acting separately:

$$q_p = \sum_1^N s_{0i}\sin(i\omega t + \varphi_{Fi} + \varphi_i)$$

where

$$s_{0i} = \frac{F_{0i}}{k\left\{\left[1-(i\eta)^2\right]^2 + (2ib_r\eta)^2\right\}^{1/2}}$$

$$\varphi_i = \operatorname{arc}tg\frac{2b_r i\eta}{1-(i\eta)^2}$$

$$F_{0i} = \sqrt{F_{1i}^2 + F_{2i}^2};\quad \varphi_i = arctg\frac{F_{1i}}{F_{2i}} \qquad \omega = \frac{2\pi}{T_F}$$

The general displacement is given by:

$$q = q_0 + q_h + q_p \qquad q_0 = \frac{F_{10}}{k}$$

The kinematics' (seismic) excitation.
The exciting considered so far was caused by the force acting on the moving mass. Now we shall consider that the frame moves harmonically according to the following formula:

$$q_z = h\sin\omega t$$

The differential equation of motion will be

$$\ddot{q} + 2\delta\dot{q} + \Omega_0^2 q = p_0\sin(\omega t + \varphi_z)$$

where

$$p_0 = \Omega_0^2 h\sqrt{1+(2b_r\eta)^2};\quad \varphi_z = arctg(2b_r\eta)$$

The particular solution will be $q_p = s_0\sin(\omega t + \varphi_z - \varphi)$ and

$$s_0 = \frac{h\sqrt{1+(2b_r\eta)^2}}{\sqrt{(1-\eta^2)^2 + (2b_r\eta)^2}} \quad or \quad \lambda = \frac{s_0}{h} = \frac{\sqrt{1+(2b_r\eta)^2}}{\sqrt{(1-\eta^2)^2 + (2b_r\eta)^2}}$$

$$\varphi = arctg\frac{2b_r\eta^3}{1=\eta^2 + (2b_r\eta)^2}$$

3.2 Vibrations of n-degree-of-freedom systems

The equation of motion is given by the matrix equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}(t)$$

M represents the mass matrix**, B –** the damping matrix, **K –** the stiffness matrix, and **Q**(t) stands for the vector of exciting forces.

3.2.1 Free vibrations

We start by solution of free vibrations described by the equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

For real systems the matrices **M** and **K** are symmetric, considering that **q** is harmonic.

$$\mathbf{q} = \mathbf{u}e^{i\Omega t}$$

When applying the given expression, we can get

$$\left(\mathbf{K} - \Omega^2\mathbf{M}\right)\mathbf{u} = \mathbf{0}$$

For non-trivial solution ($\mathbf{u} \neq \mathbf{0}$) the following must be true: $\mathbf{K} - \Omega_0^2\mathbf{M} = \mathbf{0}$. While the system of equations is homogeneous the determinant of the system must be

$$\det\left|\mathbf{K} - \Omega_0^2\mathbf{M}\right| = 0$$

This is the characteristic equation, which is also called the *frequency equation*. From here we determine the natural frequencies $\Omega_{01} \le \Omega_{02} \le ..... \le \Omega_{0n}$. This way we obtain the *modal eigenvectors*, or shortly *modal vectors*:

$$\mathbf{v}_r^T = \left[\frac{u_{r1}}{u_{r1}}, \frac{u_{r2}}{u_{r1}}, ..., \frac{u_{rn}}{u_{r1}}\right].$$

Each modal vector can be normalized by one of following expressions

$$\mathbf{v}_r^T\mathbf{v}_r = 1 \qquad \text{or}$$
$$\mathbf{v}_r^T\mathbf{M}\mathbf{v}_r = 1 \qquad \text{or}$$
$$\mathbf{v}_r^T\mathbf{K}\mathbf{v}_r = 1$$

<u>*Rayleigh method*</u>:

The following can be derived from the equation $\mathbf{K} - \Omega_0^2\mathbf{M} = \mathbf{0}$:

$$\lambda_r = \Omega_{0r}^2 = \frac{\mathbf{v}_r^T K\mathbf{v}_r}{\mathbf{v}_\mathbf{r}^\mathbf{T}\mathbf{M}\mathbf{v}_\mathbf{r}} = \frac{E_p}{E_k^*}$$

$\mathbf{v}_r$ represents the natural vector of *r* mode, E_p is the potential energy, E_k^* is the kinetic energy if Ω=0.

<u>*Eigenvalue problem*</u>

The eq. (3.2.3) can be changed into:

$$\mathbf{A}\mathbf{u}_r = \mathbf{E}\lambda_r\mathbf{u}_r$$

where

$$\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$$

In mathematics, this method is called the *eigenvalue problem.* If the matrix **A** is symmetric, we can use the *Jaoobi´smethod*.

3.2.2 Damped vibrations

The damped vibration problem is governed by equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

Because it is difficult to construct the damping matrix **B**, we introduce so-called *proportional damping*, which consists of a weighted sum of the mass and stiffness matrix

$$\mathbf{B} = \alpha\mathbf{M} + \beta\mathbf{K}$$

The damping coefficients α and β will be determined from experimental vibration testing. The advantage of this notation gives the simple equation of orthogonality

$$\mathbf{v}_r^T \mathbf{B} \mathbf{v}_s = 0 \quad pro \quad r \neq s$$

We consider the solution in the form

$$\mathbf{q} = \sum_{\mathbf{r}} C_r e^{\lambda_r t} \mathbf{v}_\mathbf{r}$$

Using this expression we obtain

$$\mathbf{q} = \sum_r e^{-\delta_r t} \left(A_r \cos \Omega_r t + B_r \sin \Omega_r t \right) \mathbf{v}_\mathbf{r} \quad or \quad \mathbf{q} = \sum_r e^{-\delta t} C \sin \Omega t + \varphi)$$

Here

$$\delta_r = \frac{\alpha m_{yr} + \beta k_{yr}}{2 m_{yr}}$$

3.2.3 Forced harmonic response

The case of forced harmonic response of an n-degree of freedom oscillator is important mainly for two reasons:

1. The system response in the frequency domain is the easiest one to be identified experimentally.
2. The harmonic regime is suitable for many excitations cases encountered in the engineering practice.

The differential equation of motion is

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q}_0 e^{i\omega t}$$

The solution consists of two parts – the homogeneous and particular solution. Let us consider the particular solution $\mathbf{q} = \tilde{\mathbf{s}}_{0p} e^{i\omega t}$. By substitution in the previous equation we get

$$\tilde{\mathbf{s}}_{0p} = (\mathbf{K} - \omega^2 \mathbf{M} + i\omega \mathbf{B})^{-1} \mathbf{Q}_0$$

If the external force is a general function of time, the particular solution will be given by the Duhamel integral:

$$\mathbf{q}_p = \sum_{r=1}^{n} \frac{\mathbf{v}_r \mathbf{v}_r^T}{\Omega_r \mathbf{v}_r^T \mathbf{M} \mathbf{v}_r} \int_0^t \mathbf{Q}(\tau) e^{-\delta_r (t-\tau)} \sin \Omega_r (t-\tau) d\tau$$

3.2.4 Reduction of the number of degrees of freedom

The designer is usually not interested in all the natural frequencies with the systems with high number of degrees of freedom. In such a case we reduce the number of degrees of freedom. When using any method, it is necessary to observe the condition, that the values of the natural frequencies of the reduced system must be equal to the natural frequencies of the original system. For very large systems the *Lanczos method* is used.

Let us consider the adapted equation of motion of the free vibration:

$$\mathbf{Kv} = \lambda\mathbf{Mv}$$

The original system has n degree of freedom. Let us choose $m<n$ degree of freedom and introduce the *reducing matrix* $\mathbf{R} = [\mathbf{r}_i]$ of order (n,m). $\mathbf{r}_i$ these are so called *Lanczos vectors.* Then we introduce the vector $\mathbf{y}$ (m,1) and therefore

$$\mathbf{v} = \mathbf{Ry}$$

By substitution in the equation of motion we get

$$\mathbf{KRy} = \lambda\mathbf{MRy}$$

Multiplying this equation by $\mathbf{R}^T$ we obtain

$$(\mathbf{R}^T\mathbf{KR})\mathbf{y} = \lambda(\mathbf{R}^T\mathbf{MR})\mathbf{y}$$

The original system is reduced to the order m. The reduced matrix $\mathbf{R}$ is constructed in such a way that $\mathbf{R}^T\mathbf{MR}$ is diagonal and unitary, and the matrix $\mathbf{R}^T\mathbf{KR}$ is three-diagonal and symmetric. Then it is possible to write

$$\mathbf{R}^T\mathbf{KRy} = \lambda\mathbf{y}$$

This way, the original problem has been transformed into an eigenvalue problem of a symmetric and three-diagonal matrix of a lower order m, which can be solved by the Jacobi method.

3.3 Continuous systems

So far, vibrating systems have been regarded as an assembly of discretised elements – rigid elements with mass linked together by spring and dissipative elements, characterized by their stiffness and their damping coefficients but without mass. Such a system is an idealized one. In most cases, the main bodies are deformable, and the elastic elements, which connect the main bodies, have also their own inertia.

In order to formulate the equations of motion of the continuous system, we will apply the theory of continuum mechanics where the equations of motion are expressed in terms of displacement field $u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)$ together with the boundary conditions to be satisfied. The space variables x, y, and z being continuous, the system so described possesses an infinity of degrees of freedom.

The course deals with the solving of vibrations of bars, shafts, beams, membranes and desks. It provides also the exact analytical and approximate solutions.

4. SENSITIVITY ANALYSIS

The aim of the sensitivity analysis is to obtain quantitative information about the sensitivity of structural natural frequencies and modal vectors to variations of physical parameters such as spring stiffness, concentrated mass, distributed mass, elasticity coefficients of material and geometry of the system.

This kind of information is becoming more valuable in dynamic structural analysis for several purposes:

- to get better knowledge of the sensitivity of a structure to slight modifications;
- to obtain derivatives for the dynamic optimisation of a structure by mathematical programming methods;
- to obtain derivatives for the updating of a dynamic model in order to get numerical and experimental results;
- to obtain derivatives for spectral and modal tuning of mechanical models.

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